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AN ANALYSIS OF SERIES CONVERGENCE IN MATHEMATICAL ANALYSIS

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Abstract

This paper presents an analysis of the convergence behavior of infinite series within the scope of mathematical analysis. Through the examination of four distinct examples, the study illustrates the application of various convergence tests, including the comparison test, ratio test, alternating series test, and absolute convergence criteria. The aim is to deepen the understanding of when and why a series converges absolutely, conditionally, or diverges. This work serves as both a practical guide and a conceptual reinforcement of series convergence for students and educators in higher mathematics.

Keywords: Infinite series, convergence, absolute convergence, conditional convergence, mathematical analysis, convergence tests, numerical series.

Introduction

The theory of infinite numerical series is one of the key branches of mathematical analysis and plays an important role in both theoretical and applied fields. Determining the convergence of a series is essential not only in pure mathematics but also in various applications such as physics, engineering, economics, and computer science. Each series requires a specific type of analysis, and selecting the appropriate convergence test is critical for accurate evaluation.

This paper focuses on examining the convergence of numerical series using different tests, with an emphasis on distinguishing between absolute and conditional convergence. By analyzing four practical examples, the paper aims to demonstrate the application of these tests and to provide readers with a deeper understanding of the concept.

Problem 1. Determine for which values of α the series $\sum_{n=1}^{\infty} a_n$ converges.

We are given:

$$a_n = \left(\sqrt{n+1} - \sqrt{n}\right)^{\alpha} \cdot \ln\frac{2n+1}{2n-1}$$

Solution:

Let's simplify $\left(\sqrt{n+1} - \sqrt{n}\right)$:

$$\sqrt{n+1}-\sqrt{n}=\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}$$

As $n\to\infty$, we can approximate

$$\sqrt{n+1}+\sqrt{n}\approx 2\sqrt{n}.$$

So, $\sqrt{n+1}-\sqrt{n}\,\square\,\frac{1}{2\sqrt{n}}=0*\left(\frac{1}{\sqrt{n}}\right)$.

Let's simplify $\ln\frac{2n+1}{2n-1}$:

$$\ln\frac{2n+1}{2n-1}=\ln(1+\frac{2n+1}{2n-1}-1)=\ln(1+\frac{(2n+1)-(2n-1)}{2n-1})$$

For small x, we know that $\ln(1+x)\approx x$. Here, as $n\to\infty,\frac{2}{2n-1}\to 0$.

So, $\ln\left(1+\frac{2}{2n-1}\right)\square\frac{2}{2n-1}\square\frac{2}{2n}=\frac{1}{n}$.

Therefore, $\ln\frac{2n+1}{2n-1}=0*\left(\frac{1}{n}\right)$.

Now, substitute these approximations back into the expression for a_n:

$$a_n\,\square\left(\frac{1}{\sqrt{n}}\right)^{\alpha}\cdot\frac{1}{n}=\frac{1}{n^{\frac{\alpha}{2}}}\cdot\frac{1}{n}=\frac{1}{n^{\frac{\alpha}{2}+1}}.$$

Let $b_n=\frac{1}{n^{1+\frac{\alpha}{2}}}$.

According to the limit comparison, if $\lim\frac{a_n}{b_n}$ is a finite positive number, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

We know that the $p-$series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$.

In our case, $p=1+\frac{\alpha}{2}$.

For the series $\sum a_n$ to converge, we must have $1+\frac{\alpha}{2}>1$.

This implies $\frac{\alpha}{2}>0$, which means $\alpha>0$.

The series $\sum_{n=1}^{\infty}a_n$ converges for $\alpha>0$.

Problem 2. $\sum_{n=1}^{\infty}\frac{\cos n}{n^{\alpha}}$

Determine the values of α for which the series is:

a) Absolutely convergent.
b) Conditionally convergent.

Solution: First, we determine for which values of α the given series converges.

If we define $a_n = \frac{1}{n^\alpha}$ and $b_n = \cos n$

1)When $\alpha > 0, \{a_n\} \downarrow$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^\alpha} = 0,$

2) $B_n = \sum_{k=1}^{n} b_k = \dfrac{\cos\frac{n+1}{2} \cdot \sin\frac{n}{2}}{\sin\frac{1}{2}}$ and $|B_n| \le \dfrac{1}{\sin\frac{1}{2}}$. $\Rightarrow$According to Dirixlet's test, the series $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{\cos n}{n^\alpha}$ converges when $\alpha > 0$. When $\alpha \le 0$, this series diviges because the necessary condition for convergence is not met when $\alpha \le 0$.

Now we examine the absolute convergence of the series. $|\frac{\cos n}{n^\alpha}| \le \frac{1}{n^\alpha}$ and from the convergence of the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ for $\alpha > 1$, we obtain the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^\alpha}$ for $\alpha > 1$.

Now we showed that when $0 < \alpha \le 1$, the given series is not absolutely convergent, the series $\sum_{n=`1}^{\infty} \frac{\cos n}{n^\alpha}$ diverges.

From the inequality

$$\frac{|\cos n|}{n^\alpha} \ge \frac{\cos^2 n}{n^\alpha} = \frac{1+\cos 2n}{2n^\alpha} = \frac{1}{2n^\alpha} + \frac{\cos 2n}{2n^\alpha},$$

and the fact that the series $\sum_{n=1}^{\infty} \frac{\cos 2n}{2n^\alpha}$ converges by Dirichlet's test, and that the series x_0 diverges, we deduce, by the comparison test, that the series $\{\sum_n x_n\}$ also diverges, and the series $S_n(x)$ diverges.

Thus, the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^\alpha}$

a) Is absolutely convergent when $\alpha > 1$
b) Is conditionally convergent when $0 < \alpha \le 1$.

Problem 3. Let $\{a_n\}_{n=0}^{\infty}$ numerical sequence defined by $0 < a_0 < \pi$ and

$$a_n = \frac{1}{n} \sum_{k=0}^{n-1} arctg(a_k) \quad \text{for} \quad n \ge 1$$

Prove that the numerical sequence $\{a_n \sqrt{\ln(n)}\}$ is convergent and find its limit.

Solution:

By the recurrence relation

$$a_{n+1}=\frac{\sum_{k=0}^{n-1}arctg(a_k)+arctg(a_n)}{n+1}=\frac{na_n+arctg(a_n)}{n+1}$$

Which implies that for $0<a_0<\pi$, and note that $arctg(x)\le x$

$$0<a_{n+1}<\frac{na_n+a_n}{n+1}=a_n$$

that is sequence $(a_n)_n$ is positive and decreasing and therefore it has a limit l which satisfies $(n+1)l=nl+\sin(l)$. Thus, it follows that $l=0$. Moreover, by Taylor approximation,

$$a_{n+1}=\frac{na_n+a_n-\frac{a_n^{\ 3}}{3}+0(a_n^3)}{n+1}=a_n\left(1-\frac{a_n^2}{3(n+1)}+\frac{0(a_n^2)}{n+1}\right)$$

and

$$\frac{1}{a_{n+1}^2}=\frac{1}{a_n^2}\left(1-\frac{a_n^2}{3(n+1)}+\frac{0(a_n^2)}{n+1}\right)^{-2}=\frac{1}{a_n^2}\left(1+\frac{2a_n^2}{3(n+1)}+\frac{0(a_n^2)}{n+1}\right)=\frac{1}{a_n^2}+\frac{2}{3(n+1)}+\frac{0(1)}{n+1}$$

Finally, by the Stolz-Cesaro Theorem,

$$\lim_{n\to\infty}\left(a_n\sqrt{\ln(n)}\right)^2=\lim_{n\to\infty}\frac{\ln(n)}{\frac{1}{a_n^{\ 2}}}\Rightarrow\lim_{n\to\infty}\frac{\ln(n+1)-\ln(n)}{\frac{1}{a_{n+1}^2}-\frac{1}{a_n^{\ 2}}}=\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{2}{3(n+1)}+\frac{0(1)}{n+1}}=\frac{3}{2}$$

Since, $a_n\sqrt{\ln(n)}\ge 0$, we may conclude that $\lim_{n\to\infty}a_n\sqrt{\ln(n)}=\sqrt{\frac{3}{2}}$.

Problem 4. Compute the sum

$$\sum_{k=1}^{n}\tan^2\left(\frac{\pi}{2n+1}\right).$$

Solution:

Consider $P_n(x)=(x^2+1)^{n+\frac{1}{2}}\cdot\sin(2n-1)\arctan(x)$. Let's show that $P_n(x)$ is a polynomial.

Notice that

$$2i\sin(n\alpha)=e^{in\alpha}-e^{-in\alpha}=(\cos(\alpha)+i\sin(\alpha))^n-(\cos(\alpha)-i\sin(\alpha))^n=$$

$$\sum_{k=0}^{n}\binom{n}{k}\cos^k\alpha\sin^{n-k}\alpha\left(i^{n-k}-(-i)^{n-k}\right)$$

$$\sum_{k=0}^{n}\binom{n}{k}\cos^k\alpha\sin^{n-k}\alpha\left(e^{\frac{\pi i(n-k)}{2}}-e^{\frac{\pi i(n-k)}{2}}\right)=2i\sin^n\alpha\sum_{k=0}^{n}\binom{n}{k}\cot^k\alpha\sin\left(\frac{\pi(n-k)}{2}\right)$$

Hence

$$P_n(x)=(x^2+1)^{n+1/2}\cdot\sin^{2n+1}(\arctan x)\sum_{k=0}^{2n+1}\binom{2n+1}{k}\cot^k(\arctan x)\sin\left(\frac{\pi(2n+1-k)}{2}\right)$$

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$$= x^{2n+1}\sum_{k=0}^{2n+1}\binom{2n+1}{k}\frac{1}{x^k}\sin\left(\frac{\pi(2n+1-k)}{2}\right)$$

$$=(-1)^n x^{2n+1}+(-1)^{n-1}\binom{2n+1}{2}x^{2n-1}+\ldots+\binom{2n+1}{2n}x.$$

Denote $x_k=\frac{\tan\pi}{2n+1}, k=-n,\ldots,n.$ Clearly $x_{-n},\ldots,x_n$ are roots of P_n and as $\deg P_n=2n+1$, it has no other roots.By Vieta's theorem $\sum_{i=-n}^{n}x_i=0$ and $\sum_{-n\le i\le n}x_ix_j=-\binom{2n+1}{2}$.

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