BOUNDARY VALUE PROBLEM FOR A PARTIAL DERIVATIVE EQUATION OF PARABOLIC TYPE WITH TWO LINES OF DEGENERACY

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Abstract:

This paper examines the edge problem for a partial differential equation of parabolic type with two lines of degeneracy, a representation of the solution is obtained, an a priori estimate of the solution is derived, theorems are obtained that prove the uniqueness and conditional stability on the correctness set of the solution.

Keywords: ill-posed problem, a priori estimate, correctness set, uniqueness, conditional stability.

КРАЕВАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ПАРАБОЛИЧЕСКОГО ТИПА С ДВУМЯ ЛИНИЯМИ ВЫРОЖДЕНИЯ

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Аннотация
В данной работе рассматривается краевая задача для уравнения в частных производных параболического типа с двумя линиями вырождения, получено представление решения, выведена априорная оценка решения, получены теоремы, доказывающие единственность и условную устойчивость на множестве корректности решения.

Ключевые слова: некорректная задача, априорная оценка, множество корректности, единственность, условная устойчивость.

Introduction

The work is devoted to the study of an ill-posed boundary value problem for equations of parabolic type with two lines of degeneracy.

Correct boundary value problems for such types of equations were considered in the works of MS Baouendi and P. Grisvard, CD Pagani and G. Talenti [1], S.A. Tersenov [2], N.N. Kislova, A.A. Kerefova, I.E. Egorova [3], S.G. Pyatkova [4]. In which the correctness of the problem for equations of the shifted and mixed-composite type was studied.

Problems that are ill-posed in the sense of J. Hadamard were studied in the works of E.M. Landisa, S.G. Kreina, M.M. Lavrentieva [5], H.A. Levine, I.E. Egorova and V.E. Fedorova, A.I. Kozhanova, S.G. Pyatkova, A.L. Bukhheim, K.S. Fayazova [6], M.Kh. Alaminova, I.O. Khazhieva [7], Y.K. Khudaiberganov [8], in which for equations of the shifted and mixed-composite type they were proven based on the idea of A.N. Tikhonov, theorems on uniqueness and conditional stability on the corresponding correctness sets.

Let $\Omega=\Omega_1\times\Omega_2\times Q$, Where $\Omega_1=\{-1<x<1,\ x\neq 0\}$, $\Omega_2=\{-1<y<1,\ y\neq 0\}$, $Q=\{0<t<T,\ T<\infty\}$.

Task. Need to find a function $u(x,y,t)$ related in the area Ω with equation

$$u_t(x,y,t)+sgn(x)u_{xx}(x,y,t)+sgn(y)u_{yy}(x,y,t)+Au(x,y,t)=0, \tag{1}$$

and satisfying the following conditions:

initial

$$u(x,y,t)\big|_{t=0}=\varphi(x,y),\ \ (x,y)\in[-1;1]^2, \tag{2}$$

borderline

$$\begin{aligned} &u_x(x,y,t)\big|_{\partial\Omega_1}=0,(y,t)\in[-1;1]\times\bar{Q},\\ &u_y(x,y,t)\big|_{\partial\Omega_2}=0,(x,t)\in[-1;1]\times\bar{Q}, \end{aligned} \tag{3}$$

and gluing conditions

$$\begin{aligned} &\left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=-0}=(-1)^i\left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=+0},\ (y,t)\in[-1;1]\times\bar{Q},\\ &\left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=-0}=(-1)^i\left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=+0},\ (x,t)\in[-1;1]\times\bar{Q}, \end{aligned} \tag{4}$$

Where, A - some constant, $(i=\overline{0,1})$, $\varphi(x,y)$ -given sufficiently smooth function, and $\varphi_x(x,y)\big|_{\partial\Omega_1}=0,\ \varphi_y(x,y)\big|_{\partial\Omega_2}=0.$

Spectral task.

Find such values λ in which the next task

$$sgn(x)\vartheta_{xx}(x,y)+sgn(y)\vartheta_{yy}(x,y)+\lambda\vartheta(x,y)=0,(x,y)\in(-1;1)^2,x,y\neq 0, \tag{5}$$

$$\begin{aligned} &\vartheta_x(x,y)\big|_{x=\pm1}=0,\ \ y\in[-1;1],\ \vartheta_y(x,y)\big|_{y=\pm1}=0,\ \ x\in[-1;1]\\ &\left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=-0}=(-1)^i\left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=+0},\ y\in[-1;1],\\ &\left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=-0}=(-1)^i\left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=+0},\ x\in[-1;1],\ \left(i=\overline{0,1}\right), \end{aligned} \tag{6}$$

We will look for a solution to problem (5), (6) using the Fourier method, assuming

$$\vartheta(x,y) = X(x)Y(y). \quad (7)$$

From conditions (6) we obtain:

$$X'(-1) = X'(+1) = 0,$$
$$X(-0) = X(+0),$$
$$X'(-0) = -X'(+0), \quad (8)$$

$$Y'(-1) = Y'(+1) = 0,$$
$$Y(-0) = Y(+0),$$
$$Y'(-0) = -Y'(+0). \quad (9)$$

Let's find the second partial derivatives of function (7):

$$\vartheta_{xx}(x,y) = X''(x)Y(y), \quad \vartheta_{yy}(x,y) = X(x)Y''(y).$$

Substituting into (5) and separating the variables, we get:

$$\frac{sgn(x)X''(x)}{X(x)} + \frac{sgn(y)Y''(y)}{Y(y)} = -\lambda.$$

Next, since $\frac{sgn(x)X''(x)}{X(x)}$ does not depend on y, A $\frac{sgn(y)Y''(y)}{Y(y)}$ from x.
Thus, we have

$$\frac{sgn(x)X''(x)}{X(x)} = -\lambda_1, \quad \frac{sgn(y)Y''(y)}{Y(y)} = -\lambda_2.$$

As a result, to find functions $X(x), Y(y)$ we get the equations:

$$sgn(x)X''(x) = -\lambda_1 X(x), \quad (9)$$

$$sgn(y)Y''(y) = -\lambda_2 Y(y). \quad (10)$$

Let us consider equations (9), (10) with the corresponding conditions (8), (9)

$$sgn(x)X''(x) = -\lambda_1 X(x),$$
$$X'(-1) = X'(+1) = 0,$$
$$X(-0) = X(+0),$$
$$X'(-0) = -X'(+0), \quad \text{(eleven)}$$

$$sgn(y)Y''(y) = -\lambda_2 Y(y),$$
$$Y'(-1) = Y'(+1) = 0,$$
$$Y(-0) = Y(+0),$$
$$Y'(-0) = -Y'(+0). \quad (12)$$

Thus, the solutions to problems (11) and (12) have the form:

at $\lambda_1 > 0, \ \lambda_2 > 0,$

$$X_k^{(1)}(x)=\begin{cases}\cos\mu_k(x-1)/\cos\mu_k \quad ,0<x<1,\\ ch\mu_k(x+1)/ch\mu_k,-1<x<0,\end{cases} k\in N,$$

$$Y_l^{(1)}(y)=\begin{cases}\cos\sigma_l(y-1)/\cos\sigma_l,0<y<1,\\ ch\sigma_l(y+1)/ch\sigma_l,-1<y<0,\end{cases} l\in N,$$

at $\lambda_1<0, \lambda_2<0,$

$$X_k^{(2)}(x)=\begin{cases}ch\mu_k(x-1)/ch\mu_k,0<x<1,\\ \cos\mu_k(x+1)/\cos\mu_k,\ -1<x<0,\end{cases} \quad k\in N,$$

$$Y_l^{(2)}(y)=\begin{cases}ch\sigma_l(y-1)/ch\sigma_l \ ,0<y<1,\\ \cos\sigma_l(y+1)/\cos\sigma_l \ ,-1<y<0,\end{cases} \quad l\in N.$$

at $\lambda_1=0, \lambda_2=0,$

$$X_0(x)=\begin{cases}1/\sqrt{2}, & 0\le x\le 1,\\ 1/\sqrt{2}, & -1\le x\le 0,\end{cases}$$

$$Y_0(y)=\begin{cases}1/\sqrt{2}, & 0\le y\le 1,\\ 1/\sqrt{2}, & -1\le y\le 0,\end{cases}$$

Where $\lambda_{1k}=\mu_k^2\ge 0,\ \lambda_{1k}=-\mu_k^2\le 0,\ \ \lambda_{2l}=\sigma_l^2\ge 0,\ \lambda_{2l}=-\sigma_l^2\le 0.$

Eigenvalues

$$\lambda_{k,l}^{(1)}=\mu_k^2+\sigma_l^2, \qquad \lambda_{k,l}^{(2)}=\mu_k^2-\sigma_l^2,$$
$$\lambda_{k,l}^{(4)}=-\mu_k^2+\sigma_l^2, \quad \lambda_{k,l}^{(5)}=-\mu_k^2-\sigma_l^2,$$

correspond to their own functions

$$\vartheta_{k,l}^{(1)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(1)}(y), \quad \vartheta_{k,l}^{(2)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(2)}(y),$$
$$\vartheta_{k,l}^{(3)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(1)}(y), \qquad \vartheta_{k,l}^{(4)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(2)}(y),$$

Where $k,l=0,1,2,....$

In both cases μ_k,σ_l- are non-negative roots of the transcendental equation $tg\alpha=-th\alpha.$ Let $\|u\|^2=(u,u)$, Where $(u,\upsilon)=\int_{-1}^{1}\int_{-1}^{1}u\upsilon dxdy$ scalar product. Besides

$$\left(\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(n)}(x,y)\right)=0, m\ne n,\ (m,n=\overline{1,4}),$$

$$\left(\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(m)}(x,y)\right)=\begin{cases}1, \ k=i\wedge l=j,\\ 0, \ k\ne i\wedge l\ne j,\end{cases} \quad (m=\overline{1,4}),$$

Where $k,l,i,j\in N.$

Let

$$\|u(x,y,t)\|^2=\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(1)}(x,y)\right)\right|^2+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(2)}(x,y)\right)\right|^2$$
$$+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k}^{(3)}(x,y)\right)\right|^2+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(4)}(x,y)\right)\right|^2 \quad (13)$$

According to the Hilbert-Schmidt theorem [9], the eigenfunctions of problem (5)-(6) form a Riesz basis in $L_2(-1;1)^2$.

A priori estimate.

By a generalized solution of the boundary value problem (1) - (4) we mean the function $u(x,y,t)$, such that $u(x,y,t)\in C(L_2(-1,1)^2,Q)$ And

$$\int_0^T\int_{-1}^1\int_{-1}^1 u(x,y,t)\left(sgn(x)sgn(y)V_t(x,y,t)-sgn(y)V_{xx}(x,y,t)-sgn(x)V_{yy}(x,y,t)\right)dxdydt=$$
$$-\int_{-1}^1\int_{-1}^1 sgn(x)sgn(y)V(x,y,0)\varphi(x,y)dxdy$$
$$-\int_0^T\int_{-1}^1\int_{-1}^1 sgn(x)sgn(y)V(x,y,t)f(x,y,t)dxdydt,$$

for any function $V(x,y,t)\in W_2^{2,1}\left((-1;1),(-1;1),(0;T)\right)$, satisfying the conditions $V(x,y,T)=0,\ V_x(-1,y,t)=V_x(1,y,t)=0,\ \ V_y(x,-1,t)=V_y(x,1,t)=0.$

Lemma 1. Let $u(x,y,t)$ is a solution to equation (1), in the region Ω and satisfies the corresponding conditions (2) - (4). Then for $u(x,y,t)$ at $t\in(0;T)$ the following inequality is true

$$\|u(x,y,t)\|\le 2\left(\|u(x,y,0)\|\right)^{1-\frac{t}{T}}\cdot\left(\|u(x,y,T)\|\right)^{\frac{t}{T}}, \quad (14)$$

The proof of this lemma can be found in [8].

Through M let us denote the correctness set defined as follows

$$M=\left\{u(x,y,t):\|u_1(x,y,T)\|_0\le m, m<\infty\right\}.$$

Uniqueness and conditional stability.

Theorem 1. If a solution to problem (1)-(4) exists and $u(x,y,t)\in M$, then the solution to problem (1)-(4) is unique.

Proof. Let equation (1) with conditions (2)-(4) have solutions $u_1(x,y,t)$ And $u_2(x,y,t)$, i.e. 2 solutions. Then the function $U(x,y,t)=u_1(x,y,t)-u_2(x,y,t)$ is a solution with zero data. For the last function, estimate (14) is true. Using the results of Lemma 1 we have $U(x,y,t)=0$ for all $(x,y,t),\ \ u_1(x,y,t)\equiv u_2(x,y,t)$.

Theorem 1 is proven.

Let $u(x,y,t)$ -solution of problem (1) - (4) with exact data, and $u_\varepsilon(x,y,t)$ - solution of problem (1) - (4) with approximate data.

Theorem 2. Let a solution to the original problem exist and $u(x,y,t),\ u_\varepsilon(x,y,t)\in M$, Besides $\|\varphi(x,y)-\varphi_\varepsilon(x,y)\|_0\le\varepsilon$. Then for the function $U(x,y,t)=u(x,y,t)-u_\varepsilon(x,y,t)$ at $t\in(0;\ T)$ the following inequality is true

$$\|U(x,y,t)\|_0\le 2(\varepsilon)^{1-\frac{t}{T}}\cdot(2m)^{\frac{t}{T}}.$$

Proof. Let the function $U(x,y,t)$ are a solution to the corresponding problem (1) - (4), and $U(x,y,0)=\varphi(x,y)-\varphi_\varepsilon(x,y)$. Function $U(x,y,t)$ satisfies the conditions of Lemma 1 and $\|U(x,y,T)\|_0^2\le 4m^2$. Then for the function $U(x,y,t)$ the following estimate is true

$$\|U(x,y,t)\|_0^2\le 4\left(\sum_{k,l=0}^{\infty}\left(\left|\varphi_{k,l}^{(1)}-\varphi_{\varepsilon l,k}^{(1)}\right|^2+\left|\varphi_{k,l}^{(2)}-\varphi_{\varepsilon l,k}^{(2)}\right|^2+\left|\varphi_{k,l}^{(3)}-\varphi_{\varepsilon l,k}^{(3)}\right|^2+\left|\varphi_{k,l}^{(4)}-\varphi_{\varepsilon l,k}^{(4)}\right|^2\right)\right)^{\frac{T-t}{T}}\times$$

$$\times\left(\sum_{k,l=0}^{\infty}\left(\left|U_{k,l}^{(1)}(T)\right|^2+\left|U_{k,l}^{(2)}(T)\right|^2+\left|U_{k,l}^{(3)}(T)\right|^2+\left|U_{k,l}^{(4)}(T)\right|^2\right)\right)^{\frac{t}{T}}\le 4\left(\varepsilon^2\right)^{1-\frac{t}{T}}\cdot\left(4m^2\right)^{\frac{t}{T}}$$

or

$$\|U(x,y,t)\|_0\le 2(\varepsilon)^{1-\frac{t}{T}}\cdot(2m)^{\frac{t}{T}}.$$

Theorem 2 is proven.

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