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INITIAL BOUNDARY-VALUE PROBLEM FOR AN INHOMOGENEOUS MIXED-TYPE PARTIAL DIFFERENTIAL EQUATION WITH TWO DEGENERATION LINES

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Abstract:
In this section, we consider a boundary-value problem for an inhomogeneous mixed-type partial differential equation with two degeneracy lines. We obtain a representation of the solution, derive an a priori estimate for the solution, and obtain theorems proving its uniqueness and conditional stability on the solution's well-posedness set.

Keywords: Mixed-type, ill-posed boundary-value problem, a priori estimates, conditional stability of the solution, uniqueness of the solution, well-posedness set.

Introduction

НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ НЕОДНОРОДНОГО УРАВНЕНИЯ СМЕШАННОГО ТИПА С ДВУМЯ ЛИНИЯМИ ВЫРОЖДЕНИЯ

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Аннотация
В этом параграфе рассматривается краевая задача для неоднородного уравнения в частных производных смешанного типа с двумя линиями вырождения, получено представление решения, выведена априорная оценка решения, получены теоремы, доказывающие единственность и условную устойчивость на множестве корректности решения.

Ключевые слова: смешанного типа, некорректная краевая задача, априорные оценки, условная устойчивость решения, единственность решения, множество корректности.

Introduction

The theory of boundary value problems for equations of mixed type is one of the most important areas of the theory of partial differential equations in mathematical physics.

In most cases, boundary value problems for equations of mixed type are well-posed. The origins of the theory of such problems are associated with the classical works of Tricomi [1] and Gellerstedt, who first posed and studied boundary value problems for equations of mixed type with a single degeneracy line.

Important applied aspects of this problem were investigated by F. I. Frankl [2]. In subsequent years, various problems for equations of mixed type were considered in the works of O. S. Ryzhov, L. D. Pilia and V. P. Fedorov, E. G. Shifrin, G. G. Cherny, and A. G. Kuzmin [3]. It should be noted that the works of F. I. Frankl, A. V. Bitsadze and K. I. Babenko laid the foundations of the modern theory of equations of mixed type and contributed to the development of research into new boundary value problems for such equations. Later, various boundary value problems for equations of mixed type were studied by many authors, in particular by V. F. Volkodavov, V. N. Vragov [4], V. I. Zhegalov, T. D. Dzhuraev, T. Sh. Kalmenov, A. I. Kozhanov, Yu. M. Krikunov, O. A. Ladyzhenskaya, M. E. Lerper, V. P. Mikhailov, E. I. Moiseev, A. M. Nakhushev [5], S. M. Ponomarev, S. P. Pulkin, K. B. Sabitov, M. S. Salakhitdinov, M. M. Smirnov, A. P. Soldatov, L. I. Chibrikova, R. S. Khairullin, B. N. Burmistrov, as well as foreign researchers S. Agmon, L. Nirenberg, M. N. Protter, K. S. Moravec, P. Germain, R. Bader, P. O. Lax, R. P. Phillips, M. Schneider, G. D. Karatoprakliev, N. I. Polivanov, G. D. Dachev and others.

Boundary value problems for equations of mixed type with two lines of degeneracy are reflected in the works of M. M. Zainulabidov, V. F. Volkodavov, V. V. Azovsky, O. I. Marichev, A. M. Ezhov, N. I. Popivanov, T. B. Lomonosov, He Kan Cher, S. I. Makarov, S. S. Ismukhamedov, Zh. Oramova, M. S. Salakhitdinova, K. B. Sabitova [6], B. Islomov and other authors.

Well-posed boundary value problems for various non-classical equations were investigated in the works of A. V. Bitsadze, S. A. Tersenov, V. N. Vragov, A. M. Nakhushev [5], and other authors. Problems for such equations were the subject of research by N. Kislov, S. G. Pyatkov [7, 8], A. I. Kozhanov [9], K. B. Sabitov [6], A. A. Gimaltinova [10], and other scientists.

Incorrect boundary value problems were studied by a number of well-known researchers, including A. L. Bukhgeim [11], V. Isakov, M. Klibanov, and K. S. Fayazov. The works of K. S. Fayazov [12], K. S., I. O. Khadzhiev [13], as well as K. S. Fayazov and Yu. K. Khudaiberganov [14] were devoted to the construction of approximate solutions for non-classical equations.

This paper is devoted to the study of an ill-posed boundary value problem for a second-order partial differential equation of mixed hyperbolic-elliptic type with two lines of degeneration.

Let $\Omega = \Omega_1 \times \Omega_2 \times Q$, where $\Omega_1 = \{-1 < x < 1\}$, $\Omega_2 = \{-1 < y < 1\}$, $Q = \{0 < t < T,\ T < \infty\}$.

Problem. It is required to find a function $u(x,y,t)$ satisfying in the domain Ω the equation

$$u_{tt}(x,y,t) + sgn(x)u_{xx}(x,y,t) + sgn(y)u_{yy}(x,y,t) + au(x,y,t) = f(x,y,t), \qquad (1)$$

and satisfying the following conditions:

initial

$$\left.\frac{\partial^i u(x,y,t)}{\partial t^i}\right|_{t=0} = \varphi_i(x,y), (x,y) \in [-1;1]^2, \qquad (2)$$

boundary

$$\begin{aligned} &u_x(x,y,t)\big|_{\partial\Omega_1} = 0, (y,t) \in [-1;1] \times \bar{Q}, \\ &u_y(x,y,t)\big|_{\partial\Omega_2} = 0, (x,t) \in [-1;1] \times \bar{Q}, \end{aligned} \qquad (3)$$

and gluing conditions

$$\begin{aligned} &\left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=-0} = (-1)^i \left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=+0}, \ (y,t) \in [-1;1] \times \bar{Q}, \\ &\left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=-0} = (-1)^i \left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=+0}, \ (x,t) \in [-1;1] \times \bar{Q}, \end{aligned} \qquad (4)$$

where $\left(i=\overline{0,1}\right)$, $\varphi_0(x,y)$, $\varphi_1(x,y)$ and $f(x,y,t)-$are given sufficiently smooth functions, with $\varphi_i(x,y)\big|_{\partial\Omega_0}=0, f(x,y,t)\big|_{\partial\Omega_0}=0.$

In this paper, a boundary value problem for an inhomogeneous partial differential equation of mixed type with two lines of degeneration is considered; a representation of the solution is obtained, an a priori estimate of the solution is derived, and theorems proving uniqueness and conditional stability on the set of correctness of the solution are obtained.

Spectral problem. Find such values λ for which the following problem

$$sgn(x)\vartheta_{xx}(x,y)+sgn(y)\vartheta_{yy}(x,y)+\lambda\vartheta(x,y)=0,(x,y)\in(-1;1)^2,x,y\neq 0, \quad (5)$$

$$\vartheta_x(x,y)\big|_{x=\pm1}=0,\ \ y\in[-1;1],\ \vartheta_y(x,y)\big|_{y=\pm1}=0,\ \ x\in[-1;1]$$

$$\left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=-0}=(-1)^i\left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=+0},\ y\in[-1;1], \quad (6)$$

$$\left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=-0}=(-1)^i\left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=+0},\ x\in[-1;1],\ \left(i=\overline{0,1}\right),$$

We shall seek the solution of problem (5), (6) by the Fourier method, setting

$$\vartheta(x,y)=X(x)Y(y). \quad (7)$$

From conditions (6) we obtain:

$$X'(-1)=X'(+1)=0,$$
$$X(-0)=X(+0), \quad (8)$$
$$X'(-0)=-X'(+0),$$
$$Y'(-1)=Y'(+1)=0,$$
$$Y(-0)=Y(+0), \quad (9)$$
$$Y'(-0)=-Y'(+0).$$

Find the second partial derivatives of function (7):

$$\vartheta_{xx}(x,y)=X''(x)Y(y),\quad \vartheta_{yy}(x,y)=X(x)Y''(y).$$

Substituting into (5) and separating the variables, we obtain:

$$\frac{sgn(x)X''(x)}{X(x)}+\frac{sgn(y)Y''(y)}{Y(y)}=-\lambda.$$

Further, since $\frac{sgn(x)X''(x)}{X(x)}$ does not depend on y, and $\frac{sgn(y)Y''(y)}{Y(y)}$ does not depend on x.

Thus, we have

$$\frac{sgn(x)X''(x)}{X(x)} = -\lambda_1, \quad \frac{sgn(y)Y''(y)}{Y(y)} = -\lambda_2.$$

As a result, to determine the functions $X(x), Y(y)$ we obtain the equations:

$$sgn(x)X''(x) = -\lambda_1 X(x), \quad (9)$$

$$sgn(y)Y''(y) = -\lambda_2 Y(y). \quad (10)$$

Consider equations (9), (10) with the corresponding conditions (8), (9)

$$\begin{aligned} & sgn(x)X''(x) = -\lambda_1 X(x), \\ & X'(-1) = X'(+1) = 0, \\ & X(-0) = X(+0), \\ & X'(-0) = -X'(+0), \end{aligned} \quad (11)$$

$$\begin{aligned} & sgn(y)Y''(y) = -\lambda_2 Y(y), \\ & Y'(-1) = Y'(+1) = 0, \\ & Y(-0) = Y(+0), \\ & Y'(-0) = -Y'(+0). \end{aligned} \quad (12)$$

Thus, the solutions of problems (11) and (12) have the form:

for $\lambda_1 > 0, \lambda_2 > 0,$

$$X_k^{(1)}(x) = \begin{cases} \cos\mu_k(x-1)/\cos\mu_k \quad ,0 < x < 1, \\ ch\mu_k(x+1)/ch\mu_k, -1 < x < 0, \end{cases} k \in N,$$

$$Y_l^{(1)}(y) = \begin{cases} \cos\sigma_l(y-1)/\cos\sigma_l, 0 < y < 1, \\ ch\sigma_l(y+1)/ch\sigma_l, -1 < y < 0, \end{cases} l \in N,$$

for $\lambda_1 < 0, \lambda_2 < 0,$

$$X_k^{(2)}(x) = \begin{cases} ch\mu_k(x-1)/ch\mu_k, 0 < x < 1, \\ \cos\mu_k(x+1)/\cos\mu_k, -1 < x < 0, \end{cases} k \in N,$$

$$Y_l^{(2)}(y) = \begin{cases} ch\sigma_l(y-1)/ch\sigma_l \ , 0 < y < 1, \\ \cos\sigma_l(y+1)/\cos\sigma_l \ , -1 < y < 0, \end{cases} l \in N.$$

for $\lambda_1 = 0, \lambda_2 = 0,$

$$X_0(x) = \begin{cases} 1/\sqrt{2}, \quad 0 \le x \le 1, \\ 1/\sqrt{2}, \ -1 \le x \le 0, \end{cases}$$

$$Y_0(y) = \begin{cases} 1/\sqrt{2}, \quad 0 \le y \le 1, \\ 1/\sqrt{2}, \ -1 \le y \le 0, \end{cases}$$

where $\lambda_{1k}=\mu_k^2\ge 0,\ \lambda_{1k}=-\mu_k^2\le 0,\ \lambda_{2l}=\sigma_l^2\ge 0,\ \lambda_{2l}=-\sigma_l^2\le 0.$

The eigenvalues

$$\lambda_{k,l}^{(1)}=\mu_k^2+\sigma_l^2,\qquad \lambda_{k,l}^{(2)}=\mu_k^2-\sigma_l^2,$$
$$\lambda_{k,l}^{(4)}=-\mu_k^2+\sigma_l^2,\quad \lambda_{k,l}^{(5)}=-\mu_k^2-\sigma_l^2,$$

correspond to the eigenfunctions

$$\vartheta_{k,l}^{(1)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(1)}(y),\quad \vartheta_{k,l}^{(2)}(x,y)=X_k^{(1)}(x)\cdot Y_l^{(2)}(y),$$
$$\vartheta_{k,l}^{(3)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(1)}(y),\qquad \vartheta_{k,l}^{(4)}(x,y)=X_k^{(2)}(x)\cdot Y_l^{(2)}(y),$$

where $k,l=0,1,2,\ldots.$

In both cases μ_k,σ_l – are nonnegative roots of the transcendental equation $tg\alpha=-th\alpha.$ Let $\|u\|^2=(u,u)$, where $(u,\upsilon)=\int_{-1}^{1}\int_{-1}^{1}u\upsilon\,dxdy$ is the scalar product. In addition

$$\left(\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(n)}(x,y)\right)=0,\ m\ne n,\ (m,n=\overline{1,4}),$$
$$\left(\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(m)}(x,y)\right)=\begin{cases}1,\ k=i\wedge l=j,\\ 0,\ k\ne i\wedge l\ne j,\end{cases}\quad (m=\overline{1,4}),$$

where $k,l,i,j\in N.$

Let

$$\|u(x,y,t)\|^2=\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(1)}(x,y)\right)\right|^2+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(2)}(x,y)\right)\right|^2$$
$$+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k}^{(3)}(x,y)\right)\right|^2+\sum_{k,l=0}^{\infty}\left|\left(u(x,y,t),\vartheta_{k,l}^{(4)}(x,y)\right)\right|^2. \qquad (13)$$

According to the Hilbert-Schmidt theorem [15] the eigenfunctions of problem (5)-(6) form a Riesz basis in $L_2(-1;1)^2$.

A priori estimate.

Theorem 1. For any generalized solution of problem (1)-(4), for $t\in(0,T)$ the following inequality holds

$$\int_0^t\|u(x,y,\tau)\|^2d\tau\le 4q(t)\left(T\|u(x,y,0)\|^2+\alpha\right)^{1-p(t)}\left(\int_0^T\|u(x,y,t)\|^2dt+\alpha\right)^{p(t)}, \qquad (14)$$

where

$$\alpha=(2T^2+1)\int_0^T\left\|f(x,y,t)\right\|^2dt+2T\left\|\varphi_0\right\|^2+\left\|\varphi_0\right\|^2+(2T+1)\left\|\varphi_1\right\|^2,$$

$$p(t)=\frac{1-e^{-2t}}{1-e^{-2T}},\ q(t)=\exp\left(\frac{2T+1}{2}\frac{\left(1-e^{-2t}\right)T-\left(1-e^{-2T}\right)t}{1-e^{-2T}}\right).$$

Proof. We seek the solution of problem (1) - (4), in the form of a series

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(u_{k,l}^{(1)}(t)\vartheta_{k,l}^{(1)}(x,y)+u_{k,l}^{(2)}(t)\vartheta_{k,l}^{(2)}(x,y)+\right.$$
$$\left.+u_{k,l}^{(3)}(t)\vartheta_{k,l}^{(3)}(x,y)+u_{k,l}^{(4)}(t)\vartheta_{k,l}^{(4)}(x,y)\right)$$

where $\left\{\vartheta_{k,l}^{(j)}(x,y)\right\}_{k,l=1}^{\infty},\left(j=\overline{1,4.}\right)$ are eigenfunctions of problem (5) - (6),

$$u_{k,l}^{(j)}(t)=(u(x,y,t),\vartheta_{k,l}^{(j)}(x,y)),\ \left(j=\overline{1,4}\right),$$

$$\varphi_{ik,l}^{(j)}=(\varphi_i(x,y),\vartheta_{k,l}^{(j)}(x,y)),\ \left(j=1,4\right),\left(i=\overline{0,1}\right),$$

$$f_{k,l}^{(j)}(t)=(f(x,y,t),\vartheta_{k,l}^{(j)}(x,y)),\ \left(j=\overline{1,4}\right),$$

Then the functions $u_{k,l}^{(j)}(t),\left(j=\overline{1,4}\right)$ are solutions of a differential equation of the form

$$(u_{k,l}^{(j)}(t))_{tt}-(\lambda_{k,l}^{(j)}-a)u_{k,l}^{(j)}(t)=f_{k,l}^{(j)}(t),$$
$$u_{k,l}^{(j)}(0)=\varphi_{1k,l}^{(j)},\ (u_{k,l}^{(j)}(0))_t=\varphi_{1k,l}^{(j)},\ k,l\in N.$$

It is easy to see that for $u_{k,l}^{(j)}(t)$ the following equalities hold

$$u_{k,l}^{(j)}(t)=\begin{cases}\dfrac{1}{\sqrt{\lambda_{k,l}^{(j)}-a}}\displaystyle\int_0^t f_{k,l}^{(j)}(\tau)sh\sqrt{(\lambda_{k,l}^{(i)}-a)}(t-\tau)d\tau+\varphi_{0k,l}^{(j)}ch\sqrt{(\lambda_{k,l}^{(j)}-a)}t+\\ \qquad+\dfrac{\varphi_{1k,l}^{(j)}sh\sqrt{(\lambda_{k,l}^{(i)}-a)}t}{\sqrt{(\lambda_{k,l}^{(i)}-a)}},\ (\lambda_{k,l}^{(i)}-a)>0,\ (j=\overline{1,3}),\\ \displaystyle\int_0^t(t-\tau)f_{k,l}^{(j)}(\tau)d\tau+\varphi_{1k,l}^{(j)}t+\varphi_{0k,l}^{(j)},\ \lambda_{k,l}^{(j)}-a=0,\ (j=\overline{2,3}),\\ \dfrac{1}{\sqrt{-(\lambda_{k,l}^{(i)}-a)}}\displaystyle\int_0^t f_{k,l}^{(j)}(\tau)\sin\sqrt{-(\lambda_{k,l}^{(i)}-a)}(t-\tau)d\tau+\varphi_{0k,l}^{(j)}\cos\sqrt{-(\lambda_{k,l}^{(i)}-a)}t+\\ \qquad+\dfrac{\varphi_{1k,l}^{(j)}\sin\sqrt{-(\lambda_{k,l}^{(i)}-a)}t}{\sqrt{-(\lambda_{k,l}^{(i)}-a)}},\ (\lambda_{k,l}^{(i)}-a)<0,\ (j=\overline{2,4}),\ k,l=0,1,\dots.\end{cases}$$

According to (13) we have

$$\|u(x,y,t)\|^2=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\left|u_{k,l}^{(1)}(t)\right|^2+\left|u_{k,l}^{(2)}(t)\right|^2+\left|u_{k,l}^{(3)}(t)\right|^2+\left|u_{k,l}^{(4)}(t)\right|^2\right).$$

Lemma 1. Let $v(t)$ be a solution of the equation

$$v''(t)-\lambda v(t)=k(t)$$

and satisfies the conditions $v(0)=p_1$, $v'(0)=p_2$. Then for the solution of this equation for $t\in(0,T)$ the following inequality holds

$$\int_0^t v^2(\tau)d\tau\le q(t)\left(Tp_1^2+\alpha\right)^{1-p(t)}\left(\int_0^T v^2(t)dt+\alpha\right)^{p(t)},$$

where λ is some constant, $k(t)$ is a given function,

$$\alpha=(2T^2+1)\int_0^T k^2(t)dt+2\left|\lambda p_1^2-p_2^2\right|T+2\left|p_1p_2\right|,\qquad p(t)=\frac{1-e^{-2t}}{1-e^{-2T}},$$

$$q(t)=\exp\left(\frac{2T+1}{2}\frac{\left(1-e^{-2t}\right)T-\left(1-e^{-2T}\right)t}{1-e^{-2T}}\right).$$

The proof of this lemma can be found in [13].

Introduce the norm

$$\|\varphi(x,y)\|_1^2=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\lambda_{k,l}^{(1)}\left(\varphi_{k,l}^{(1)}\right)^2+\left|\lambda_{k,l}^{(2)}\right|\left(\varphi_{k,l}^{(2)}\right)^2+\left|\lambda_{k,l}^{(3)}\right|\left(\varphi_{k,l}^{(3)}\right)^2+\left|\lambda_{k,l}^{(4)}\right|\left(\varphi_{k,l}^{(4)}\right)^2\right).$$

According to Lemma 1, for the solutions of the problems at each fixed $k,l=0,1,....$ the inequalities hold

$$\int_0^t\left(u_{k,l}^{(j)}(\tau)\right)^2d\tau\le q(t)\left(T\left(u_{k,l}^{(j)}(0)\right)^2+\alpha_{k,l}^{(j)}\right)^{1-p(t)}\times$$

$$\times\left(\int_0^T\left(u_{k,l}^{(j)}(t)\right)^2dt+\alpha_{k,l}^{(j)}\right)^{p(t)},\left(j=\overline{1,4}\right)\qquad(15)$$

where

$$\alpha_{k,l}^{(j)}=(2T^2+1)\int_0^T\left(f_{k,l}^{(j)}(t)\right)^2dt+2\left|\lambda_{k,l}^{(j)}\left(\varphi_{0k,l}^{(j)}\right)^2-\left(\varphi_{1k,l}^{(j)}\right)^2\right|T+2\left|\varphi_{0k,l}^{(j)}\varphi_{1k,l}^{(j)}\right|.$$

(16)

It should be noted that from (16) one can easily derive the inequality

$$\alpha_{k,l}^{(j)} \le (2T^2+1)\int_0^T \left(f_{k,l}^{(j)}(t)\right)^2 dt + (2T\lambda_{k,l}^{(j)}+1)\left(\varphi_{0k,l}^{(j)}\right)^2 + (2T+1)\left(\varphi_{1k,l}^{(j)}\right)^2, \left(j=\overline{1,4}\right).$$

Summing inequalities (15) over $k,l=0,1,....$ and taking into account Holder's inequality, we obtain

$$\int_0^t \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left\{ \left|u_{2k,l}^{(1)}(t)\right|^2 + \left|u_{2k,l}^{(2)}(t)\right|^2 + \left|u_{2k,l}^{(3)}(t)\right|^2 + \left|u_{2k,l}^{(4)}(t)\right|^2 \right\} \right) d\tau \le$$

$$\le 4q(t)\left(T\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left(\left|u_{2k,l}^{(1)}(0)\right|^2 + \left|u_{2k,l}^{(2)}(0)\right|^2 + \left|u_{2k,l}^{(3)}(0)\right|^2 + \left|u_{2k,l}^{(4)}(0)\right|^2 \right) + \right.$$

$$\left. + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left(\left|\alpha_{k,l}^{(1)}\right| + \left|\alpha_{k,l}^{(2)}\right| + \left|\alpha_{k,l}^{(3)}\right| + \left|\alpha_{k,l}^{(4)}\right| \right) \right)^{1-p(t)} \times$$

$$\times \left(\int_0^T \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left(\left|u_{2k,l}^{(1)}(t)\right|^2 + \left|u_{2k,l}^{(2)}(t)\right|^2 + \left|u_{2k,l}^{(3)}(t)\right|^2 + \left|u_{2k,l}^{(4)}(t)\right|^2 \right) \right) dt + \right.$$

$$\left. + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left(\left|\alpha_{k,l}^{(1)}\right| + \left|\alpha_{k,l}^{(2)}\right| + \left|\alpha_{k,l}^{(3)}\right| + \left|\alpha_{k,l}^{(4)}\right| \right) \right)^{p(t)},$$

and summing the above inequalities, we finally obtain

$$\int_0^t \left\|u(x,y,\tau)\right\|^2 d\tau \le 4q(t)\left(T\left\|\varphi_0\right\|^2 + \alpha\right)^{1-p(t)} \left(\int_0^T \left\|u(x,y,t)\right\|^2 dt + \alpha \right)^{p(t)}.$$

Theorem 1 is proved.

Introduce the set of correctness M as follows

$$M = \left\{ u(x,y,t) : \int_0^T \left\|u(x,y,t)\right\|^2 dt \le m^2, m<\infty \right\}. \qquad (17)$$

Uniqueness and conditional stability.

Theorem 2. If a solution of problem (1) - (4) exists and $u(x,y,t)\in M$, then the solution of problem (1) - (4) is unique.

Proof. Let $u_1(x,y,t)$ and $u_2(x,y,t)$ be solutions of problems (1) - (4) with the same data. Then $u(x,y,t)=u_1(x,y,t)-u_2(x,y,t)$ will be a solution of problems

(1) - (4) with a homogeneous equation and zero data. The function $u(x,y,t)$ is a solution of the corresponding homogeneous equation with zero data. This equation satisfies the conditions of Theorem 1, i.e. inequality (14) holds. From inequality (14) it follows that $\|u(x,y,t)\|=0$. Therefore, for arbitrary $(x,y,t)\in\Omega$, $u(x,y,t)\equiv 0$ or $u_1(x,y,t)\equiv u_2(x,y,t)$. Theorem 2 is proved.

Let $u(x,y,t)$ be the solution of problem (1) - (4) with exact data, and $u_\varepsilon(x,y,t)$ be the solution of problem (1) - (4) with approximate data.

Theorem 3. Let $u(x,y,t), u_\varepsilon(x,y,t)\in M$ and $\|\varphi_0(x,y)-\varphi_{0\varepsilon}(x,y)\|_1\le\varepsilon$, $\|\varphi_1(x,y)-\varphi_{1\varepsilon}(x,y)\|\le\varepsilon$, $\|f(x,y,t)-f_\varepsilon(x,y,t)\|\le\varepsilon$. Then for the function $U(x,y,t)=u(x,y,t)-u_\varepsilon(x,y,t)$ the inequality holds

$$\int_0^t \|U(x,y,\tau)\|_0^2 d\tau \le 4q(t)\left\{T\varepsilon^2+\alpha_\varepsilon\right\}^{1-p(t)}\left\{4m^2+\alpha_\varepsilon\right\}^{p(t)},$$

for all $t\in(0;T)$, where $\alpha_\varepsilon=\varepsilon^2\left(2T^3+5T+2\right)$.

Proof. Let the function $U(x,y,t)$ be a solution of the corresponding problem (1) - (4), with $U(x,y,0)=\varphi_0(x,y)-\varphi_{0\varepsilon}(x,y)$, and $U_t(x,y,0)=\varphi_1(x,y)-\varphi_{1\varepsilon}(x,y)$. The function $U(x,y,t)$ satisfies the conditions of Theorem 1 and $\int_0^T \|U(x,y,t)\|_0^2 dt\le 4m^2$. Then for the function $U(x,y,t)$ the following estimate holds

$$\int_0^t \|U(x,y,\tau)\|_0^2 d\tau \le 4q(t)\left\{T\varepsilon^2+\alpha_\varepsilon\right\}^{1-p(t)}\left\{4m^2+\alpha_\varepsilon\right\}^{p(t)}.$$

Theorem 3 is proved.

References:

1. Tricomi, F. G. Linear Equations of Mixed Type. Moscow–Leningrad: Gostekhizdat, 1947.
2. Frankl, F. I. Selected Works on Gas Dynamics. Moscow: Nauka, 1973.

3. Kuzmin, A. G. Nonclassical Equations of Mixed Type and Their Applications in Gas Dynamics. Leningrad: Leningrad University Press, 1990.
4. Vragov, V. N. On the Theory of Boundary Value Problems for Equations of Mixed Type in Space. Differential Equations, 1977, Vol. 13, No. 6, pp. 1098–1105.
5. Nakhushev, A. M. Equations of Mathematical Biology. — Moscow: Vysshaya shkola, 1995.
6. Sabitov K. B. On the theory of the Frankl problem for equations of mixed type // Bulletin of the Russian Academy of Sciences. Mathematical series. — 2017. — Vol. 81, No. 1. — Pp. 99–136.
7. Pyatkov S. G. Solvability of a boundary value problem for a second-order mixed type equation // Nonclassical partial differential equations: Collection of scientific papers. — Novosibirsk, 1988. — Pp. 77–90.
8. Pyatkov S. G. On the solvability of a boundary value problem for a parabolic equation with changing time direction // Reports of the USSR Academy of Sciences. — 1985. — Vol. 285, No. 6. — Pp. 1322–1327.
9. Kozhanov A. I. Composite type equations and inverse problems. — Utrecht: VSP, 1999.
10. Gimaltdinova A. A. The Dirichlet Problem for a Mixed-Type Equation with Two Mutually Perpendicular Transition Lines in a Rectangular Domain // Bulletin of Samara State Technical University. Series: Physical and Mathematical Sciences. — 2015. — Vol. 19, No. 4. — Pp. 634–649.
11. Bukhgeim A. L. Ill-Posed Problems, Number Theory, and Visualization // Siberian Mathematical Journal. — 1992. — Vol. 33, No. 3. — Pp. 389–402.
12. Fayazov K. S. An Ill-Posed Boundary Value Problem for a Second-Order Mixed-Type Equation // Uzbek Mathematical Journal. — 1995. — No. 2. — Pp. 89–93.
13. Khadzhiev I. O. Conditional Well-Posedness and Approximate Solution of a Boundary-Value Problem for a System of Second-Order Mixed-Type Equations // Journal of the Siberian Federal University. Mathematics and Physics. - 2018. - Vol. 11, No. 2. - Pp. 231-241.
14. Fayazov K. S., Khudaiberganov Yu. K. Ill-Posed Boundary-Value Problem for a System of Partial Differential Equations with Two Lines of Degeneracy //

Journal of the Siberian Federal University. Mathematics and Physics. - 2019. - Vol. 12, No. 3. - Pp. 392-401.

15. Kolmogorov A. N., Fomin S. V. Elements of the Theory of Functions and Functional Analysis. - Moscow: Nauka, 1981. - 544 p.

16. Sharipov J. I., Islomov B. I., Xojiev T. Numerical solution of the Dirichlet problem for a stationary nonlinear system of partial differential equations // American Journal of Education and Learning. — 2026. — Vol. 4, No. 3. - DOI: 10.5281/zenodo.18907046.